11. cvičení - řešení

Příklad 1 (a) $\int_{-3}^{7} x^3 - 2x + 1 \mathrm{d}x$

$$\int_{-3}^{7} x^3 - 2x + 1 \mathrm{d}x = \left[\frac{x^4}{4} - x^2 + x\right]_{-3}^{7} = \left(\frac{7^4}{4} - 49 + 7\right) - \left(\frac{81}{4} - 9 - 3\right) = 580 - 40 + 10 = 550$$

Příklad 1 (b) $\int_0^3 |1 - x| \mathrm{d}x$
Použijeme aditivitu integrálu

$$\int_0^3 |1 - x| \mathrm{d}x = \int_0^1 1 - x \mathrm{d}x + \int_1^3 x - 1 \mathrm{d}x = \left[x - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^2}{2} - x\right]_1^3 =$$
$$= \left(1 - \frac{1}{2}\right) - \left(0 - \frac{0^2}{2}\right) + \left(\frac{9}{2} - 3\right) - \left(\frac{1}{2} - 1\right) = \frac{1}{2} - 0 + \frac{3}{2} + \frac{1}{2} = \frac{5}{2}$$

Příklad 1 (c) $\int_0^{2\pi} 2\sin^2 x \mathrm{d}x$

$$\int_0^{2\pi} 2\sin^2 x \mathrm{d}x \overset{\text{lin.}}{=} 2\int_0^{2\pi} \sin x \cdot \sin x \overset{\text{PP}}{=} 2\left[-\sin x \cos x\right]_0^{2\pi} - 2\int_0^{2\pi} -\cos^2 x \mathrm{d}x =$$
$$= 2\left[-\sin x \cos x\right]_0^{2\pi} - 2\int_0^{2\pi} -(1 - \sin^2 x) \mathrm{d}x = 2\left[-\sin x \cos x\right]_0^{2\pi} - 2\int_0^{2\pi} \sin^2 x - 1 \mathrm{d}x =$$
$$= 2\left[-\sin x \cos x\right]_0^{2\pi} - 2\int_0^{2\pi} \sin^2 x \mathrm{d}x + 2\left[x\right]_0^{2\pi} = 2\left[x - \sin x \cos x\right]_0^{2\pi} - \int_0^{2\pi} 2\sin^2 x \mathrm{d}x$$
$$\implies 2\int_0^{2\pi} 2\sin^2 x \mathrm{d}x = 2\left[x - \sin x \cos x\right]_0^{2\pi} \implies \int_0^{2\pi} 2\sin^2 x \mathrm{d}x = \left[x - \sin x \cos x\right]_0^{2\pi} =$$
$$= (2\pi - \sin(2\pi)\cos(2\pi)) - (0 - \sin 0 \cos 0) = 2\pi$$

Příklad 1 (d) $\int_{\frac{1}{e}}^{e} |\log x| \mathrm{d}x$

Použijeme aditivitu. Navíc platí: $\int \log x \mathrm{d}x = x\log x - x$. (známo z předchozích cvičení, lze spočíst pomocí per partes).

$$\int_{\frac{1}{e}}^{e} |\log x| \mathrm{d}x = \int_{\frac{1}{e}}^{1} -\log x \mathrm{d}x + \int_1^e \log x \mathrm{d}x = \left[x - x\log x\right]_{\frac{1}{e}}^{1} + \left[x\log x - x\right]_1^e =$$
$$= (1 - 0) - \left(\frac{1}{e} - \frac{1}{e}\log\frac{1}{e}\right) + (e\log e - e) - (0 - 1) = 1 - \frac{1}{e} + \frac{1}{e}(\log 1 - \log e) + e - e + 1 =$$
$$= 1 - \frac{1}{e} - \frac{1}{e} + 1 = 2 - \frac{2}{e}$$

Příklad 1 (e) $\int_0^{\pi} x^2 \cos^2 x \mathrm{d}x$
Výše jsme spočetli, že $\int \sin^2 x \mathrm{d}x = \frac{1}{2}(x - \sin x \cos x)$. Z toho plyne:

$$\int \cos^2 x \mathrm{d}x = \int 1 - \sin^2 x \mathrm{d}x = x - \int \sin^2 x = x - \frac{1}{2}(x - \sin x \cos x) = \frac{1}{2}(x + \sin x \cos x)$$

Získanou rovnost využijeme níže.

$$\int_0^\pi x^2\cos^2 x\mathrm{d}x \overset{\text{PP}}{=} \left[x^2\frac{1}{2}\left(x+\sin x\cos x\right)\right]_0^\pi - \int_0^\pi 2x\frac{1}{2}\left(x+\sin x\cos x\right)\mathrm{d}x =$$
$$= \left[x^2\frac{1}{2}\left(x+\sin x\cos x\right)\right]_0^\pi - \int_0^\pi x^2 + x\sin x\cos x\mathrm{d}x =$$
$$= \left[x^2\frac{1}{2}\left(x+\sin x\cos x\right) - \frac{x^3}{3}\right]_0^\pi - \int_0^\pi x\sin x\cos x\mathrm{d}x$$

Spočteme $\int x\sin x\cos x\mathrm{d}x$ pomocí per partes.

$$\int \sin x\cos x\mathrm{d}x \overset{\text{PP}}{=} -\cos^2 x - \int(-\cos x)(-\sin x)\mathrm{d}x \implies \int \sin x\cos x\mathrm{d}x \overset{\text{c}}{=} -\frac{\cos^2 x}{2}$$
$$\int x\sin x\cos x\mathrm{d}x \overset{\text{PP}}{=} -\frac{x\cos^2 x}{2} + \int\frac{\cos^2 x}{2}\mathrm{d}x \overset{\text{c}}{=} -\frac{x\cos^2 x}{2} + \frac{1}{4}\left(x+\sin x\cos x\right)$$

Dosadíme do předchozího výpočtu.

$$\left[x^2\frac{1}{2}\left(x+\sin x\cos x\right) - \frac{x^3}{3}\right]_0^\pi - \int_0^\pi x\sin x\cos x\mathrm{d}x =$$
$$= \left[\frac{x^3}{2} + \frac{x^2}{2}\sin x\cos x - \frac{x^3}{3} + \frac{x\cos^2 x}{2} - \frac{1}{4}x - \frac{\sin x\cos x}{4}\right]_0^\pi =$$
$$= \left[\frac{x^3}{6} + \frac{x^2}{2}\sin x\cos x + \frac{x\cos^2 x}{2} - \frac{1}{4}x - \frac{\sin x\cos x}{4}\right]_0^\pi = \left(\frac{\pi^3}{6} + \frac{\pi}{2} - \frac{\pi}{4}\right) - 0 = \frac{\pi^3}{6} + \frac{\pi}{4} = \frac{2\pi^3+3\pi}{12}$$

Příklad 1 (f) $\int_0^{\sqrt{3}} x\arctan x\mathrm{d}x$

$$\int_0^{\sqrt{3}} x\arctan x\mathrm{d}x \overset{\text{PP}}{=} \left[\frac{x^2}{2}\arctan x\right]_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}}\frac{x^2+1-1}{1+x^2}\mathrm{d}x =$$
$$= \left[\frac{x^2}{2}\arctan x\right]_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} 1\mathrm{d}x + \frac{1}{2}\int_0^{\sqrt{3}}\frac{1}{x^2+1}\mathrm{d}x = \left[\frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{1}{2}\arctan x\right]_0^{\sqrt{3}} =$$
$$= \left(\frac{3}{2}\arctan\sqrt{3} - \frac{\sqrt{3}}{2} + \frac{1}{2}\arctan\sqrt{3}\right) - \left(0\cdot\arctan 0 - 0 + \frac{1}{2}\arctan 0\right) = \frac{3}{2}\cdot\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{\pi}{3} - 0 =$$
$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

Příklad 1 (g) $\int_0^{\frac{\pi}{2}} e^x\sin x\mathrm{d}x$

$$\int e^x\sin x\mathrm{d}x \overset{\text{PP}}{=} -e^x\cos x + \int e^x\cos x\mathrm{d}x \overset{\text{PP}}{=\!=} -e^x\cos x + e^x\sin x - \int e^x\sin x\mathrm{d}x$$
$$\implies \int e^x\sin x\mathrm{d}x \overset{\text{c}}{=} \frac{1}{2}e^x\left(\sin x - \cos x\right)$$

$$\int_0^{\frac{\pi}{2}} e^x \sin x \mathrm{d}x = \left[\frac{1}{2}e^x\left(\sin x - \cos x\right)\right]_0^{\frac{\pi}{2}} = \frac{1}{2}e^{\frac{\pi}{2}}\left(1-0\right) - \frac{1}{2}e^0(0-1) = \frac{1}{2}e^{\frac{\pi}{2}} + \frac{1}{2}$$

Příklad 1 (h) $\int_0^{\log 4} xe^{-x}\mathrm{d}x$

$$\int xe^{-x}\mathrm{d}x \overset{\text{PP}}{=} -xe^{-x} + \int e^{-x}\mathrm{d}x \overset{\text{c}}{=} -xe^{-x} - e^{-x}$$

$$\int_0^{\log 4} xe^{-x}\mathrm{d}x = \left[-xe^{-x} - e^{-x}\right]_0^{\log 4} = -\frac{\log 4 - 1}{e^{\log 4}} - (0 - e^0) = -\frac{1}{4}(2\log 2) - \frac{1}{4} + 1 = -\frac{\log 2}{2} + \frac{3}{4}$$

Příklad 1 (i) $\int_0^{10\pi} \sqrt{1-\cos 2x}\mathrm{d}x$

$$\int \sqrt{1-\cos 2x}\mathrm{d}x = \int \sqrt{1-\left(\cos^2 x - \sin^2 x\right)}\mathrm{d}x = \int \sqrt{1-\cos^2 x + \sin^2 x}\mathrm{d}x = \int \sqrt{2\sin^2 x}\mathrm{d}x$$

Z aditivity integrálu a periodicity funkce $\sin x$ dostáváme: $\int_0^{10\pi} \sqrt{1-\cos 2x}\mathrm{d}x = 5\int_0^{2\pi} \sqrt{1-\cos 2x}\mathrm{d}x$. Dále:

$$\begin{aligned}
&\int_0^{2\pi} \sqrt{1-\cos 2x}\mathrm{d}x = \int_0^{2\pi} \sqrt{2\sin^2 x}\mathrm{d}x = \int_0^{\pi} \sqrt{2\sin^2 x}\mathrm{d}x + \int_\pi^{2\pi} \sqrt{2\sin^2 x}\mathrm{d}x = \\
&= \int_0^{\pi} \sqrt{2}\sin x\mathrm{d}x - \int_\pi^{2\pi} \sqrt{2}\sin x\mathrm{d}x = \left[-\sqrt{2}\cos x\right]_0^{\pi} + \left[\sqrt{2}\cos x\right]_\pi^{2\pi} = \\
&= \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} = 4\sqrt{2}
\end{aligned}$$

Dostáváme tedy, že $\int_0^{10\pi} \sqrt{1-\cos 2x}\mathrm{d}x = 5\int_0^{2\pi} \sqrt{1-\cos 2x}\mathrm{d}x = 5\cdot 4\sqrt{2} = 20\sqrt{2}$.

Příklad 1 (j) $\int_0^1 x^{15}\sqrt{1+3x^8}\mathrm{d}x$
V substituci níže si uvědomme, že funkce $x \mapsto 1+3x^8$ je na intervalu $(0,1)$ ryze monotónní.

$$\begin{aligned}
&\int_0^1 x^{15}\sqrt{1+3x^8}\mathrm{d}x = \left|y = 1+3x^8, \mathrm{d}y = 24x^7\mathrm{d}x, 0 \to 1+3\cdot 0^8 = 1, 1 \to 1+3\cdot 1^8 = 4\right| = \\
&= \int_1^4 \frac{y-1}{3\cdot 24}\sqrt{y}\mathrm{d}y \overset{\text{lin.}}{=} \frac{1}{72}\int_1^4 y^{\frac{3}{2}} - y^{\frac{1}{2}}\mathrm{d}y = \frac{1}{72}\left[\frac{2}{5}y^{\frac{5}{2}} - \frac{2}{3}y^{\frac{3}{2}}\right]_1^4 = \\
&= \frac{1}{72}\left(\frac{2\sqrt{4^5}}{5} - \frac{2\sqrt{4^3}}{3}\right) - \frac{1}{72}\left(\frac{2}{5} - \frac{2}{3}\right) = \frac{29}{270}
\end{aligned}$$

Příklad 1 (k) $\int_0^{\frac{\pi}{2}} \frac{1}{2\sin^2 x + 3\cos^2 x}\mathrm{d}x$
V substituci níže si uvědomme, že funkce $x \mapsto \tan x$ je na intervalu $\left(0, \frac{\pi}{2}\right)$ ryze monotónní, analogicky pro $t \mapsto t\sqrt{\frac{2}{3}}$.

$$\int_0^{\frac{\pi}{2}} \frac{1}{2\sin^2 x + 3\cos^2 x}\mathrm{d}x = \left| t = \tan x, 0 \to 0, \frac{\pi}{2} \to \infty \right| = \int_0^{\infty} \frac{1}{2\frac{t^2}{t^2+1} + 3\frac{1}{1+t^2}} \cdot \frac{1}{1+t^2}\mathrm{d}t =$$

$$= \int_0^{\infty} \frac{1+t^2}{2t^2+3} \cdot \frac{1}{1+t^2}\mathrm{d}t = \int_0^{\infty} \frac{1}{2t^2+3}\mathrm{d}t \overset{\text{lin.}}{=} \frac{1}{3}\int_0^{\infty} \frac{1}{1+\left(\frac{t\sqrt{2}}{\sqrt{3}}\right)^2}\mathrm{d}t =$$

$$= \left| u = t\sqrt{\frac{2}{3}}, \mathrm{d}u = \sqrt{\frac{2}{3}}\mathrm{d}t, 0 \to 0, \infty \to \infty \right| \overset{\text{lin.}}{=} \frac{\sqrt{3}}{3\sqrt{2}}\int_0^{\infty} \frac{1}{1+u^2}\mathrm{d}u =$$

$$= \frac{1}{\sqrt{6}}\left[\arctan u\right]_0^{\infty} = \frac{1}{\sqrt{6}}\left(\lim_{u\to\infty} \arctan u\right) - \frac{1}{\sqrt{6}} \cdot \arctan 0 = \frac{\pi}{2\sqrt{6}}$$

Příklad 1 (l) $\int_0^{2\pi} \frac{1}{(2+\cos x)(3+\cos x)}\mathrm{d}x$

Provedeme substituci $t = \tan\frac{x}{2}$, kterou je možné provést na intervalech $(-\pi + 2k\pi, \pi + 2k\pi)\,, k \in \mathbb{Z}$. Proto interval $(0, 2\pi)$, na kterém integrujeme, rozdělíme na $(0,\pi), (\pi, 2\pi)$ (díky aditivitě integrálu).

V substituci níže si uvědomme, že funkce $x \mapsto \tan\frac{x}{2}$ je na intervalu $(0,\pi)$ i na intervalu $(\pi, 2\pi)$ ryze monotónní.

$$\int_0^{\pi} \frac{1}{(2+\cos x)(3+\cos x)}\mathrm{d}x = \left| t = \tan\frac{x}{2}, 0 \to 0, \pi \to \infty \right| = \int_0^{\infty} \frac{1}{\left(2 + \frac{1-t^2}{1+t^2}\right)\left(3 + \frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\mathrm{d}t =$$

$$= \int_0^{\infty} \frac{(1+t^2)^2}{(2+2t^2+1-t^2)(3+3t^2+1-t^2)} \cdot \frac{2}{1+t^2}\mathrm{d}t = \int_0^{\infty} \frac{2+2t^2}{(t^2+3)(2t^2+4)}\mathrm{d}t =$$

$$\overset{\text{lin.}}{=} \int_0^{\infty} \frac{2}{t^2+3} - \frac{1}{t^2+2}\mathrm{d}t = \left[\frac{2}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}} - \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}\right]_0^{\infty} =$$

$$= \left(\lim_{t\to\infty} \frac{2}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}} - \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}\right) - 0 \overset{\text{VOLSF}}{=} \frac{2}{\sqrt{3}} \cdot \frac{\pi}{2} - \frac{1}{\sqrt{2}} \cdot \frac{\pi}{2} = \frac{\pi}{\sqrt{3}} - \frac{\pi}{\sqrt{8}}$$

$$\int_{\pi}^{2\pi} \frac{1}{(2+\cos x)(3+\cos x}\mathrm{d}x = \left| t = \tan\frac{x}{2}, \pi \to -\infty, 2\pi \to 0 \right| =$$

$$= \int_{-\infty}^{0} \frac{1}{\left(2 + \frac{1-t^2}{1+t^2}\right)\left(3 + \frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\mathrm{d}t =$$

$$\overset{\text{lin.}}{=} \int_{-\infty}^{0} \frac{2}{t^2+3} - \frac{1}{t^2+2}\mathrm{d}t = \left[\frac{2}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}} - \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}\right]_{-\infty}^{0} =$$

$$= 0 - \left(\lim_{t\to-\infty} \frac{2}{\sqrt{3}}\arctan\frac{t}{\sqrt{3}} - \frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}\right) = -\left(\frac{2}{\sqrt{3}} \cdot \frac{-\pi}{2} - \frac{1}{\sqrt{2}} \cdot \frac{-\pi}{2}\right) = -\left(-\frac{\pi}{\sqrt{3}} + \frac{\pi}{\sqrt{8}}\right) =$$

$$= \frac{\pi}{\sqrt{3}} - \frac{\pi}{\sqrt{8}}$$

$$\int_0^{2\pi} \frac{1}{(2+\cos x)(3+\cos x)}\mathrm{d}x = \int_0^{\pi} \frac{1}{(2+\cos x)(3+\cos x)}\mathrm{d}x + \int_{\pi}^{2\pi} \frac{1}{(2+\cos x)(3+\cos x)}\mathrm{d}x =$$

$$= \frac{2\pi}{\sqrt{3}} - \frac{\pi}{\sqrt{2}}$$

Příklad 2 (a) $\int_{-1}^{2} |x|\mathrm{d}x$.

Využijeme aditivitu integrálu ($x \mapsto |x|$ je spoj. fce v 0) a definice absolutní hodnoty, kterou tu pro pořádek připomeňme.

Zřejmě $|x| = \begin{cases} x, & x \in (0,2) \\ 0, -x, & x \in (-1,0) \end{cases}$ je spojitá funkce.

$$\int_{-1}^{2} |x| \mathrm{d}x = \int_{-1}^{0} |x| \mathrm{d}x + \int_{0}^{2} |x| \mathrm{d}x = \int_{-1}^{0} -x \mathrm{d}x + \int_{0}^{2} x \mathrm{d}x = \left[-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2} =$$
$$= 0 - \left(-\frac{(-1)^2}{2}\right) + \frac{2^2}{2} - 0 = \frac{5}{2}$$

Příklad 2 (b) $\int_{-\frac{1}{3}}^{\pi} |1-x| + |1+x| \mathrm{d}x$.

Platí: $f(x) := |1-x| + |1+x| = \begin{cases} -2x, & x < -1 \\ 2, & x \in [-1,1] \\ 2x, & x > 1 \end{cases}$ je spojitá funkce. Použijeme aditivitu integrálu.

$$\int_{-\frac{1}{3}}^{\pi} |1-x| + |1+x| \mathrm{d}x = \int_{-\frac{1}{3}}^{1} |1-x| + |1+x| \mathrm{d}x + \int_{1}^{\pi} |1-x| + |1+x| \mathrm{d}x =$$
$$= \int_{-\frac{1}{3}}^{1} 2 \mathrm{d}x + \int_{1}^{\pi} 2x \mathrm{d}x = [2x]_{-\frac{1}{3}}^{1} + \left[x^2\right]_{1}^{\pi} =$$
$$= 2 + \frac{2}{3} + \pi^2 - 1 = \frac{5}{3} + \pi^2$$

Příklad 2 (c) $\int_{\frac{1}{2}}^{\frac{3}{2}} \max\{1, x^2\} \mathrm{d}x$.

Platí: $f(x) := \max\{1, x^2\} = \begin{cases} x^2, & x < -1 \vee x > 1 \\ 1, & x \in [-1,1] \end{cases}$ je spojitá. Použijeme aditivitu integrálu.

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \max\{1, x^2\} \mathrm{d}x = \int_{\frac{1}{2}}^{1} 1 \mathrm{d}x + \int_{1}^{\frac{3}{2}} x^2 \mathrm{d}x = [x]_{\frac{1}{2}}^{1} + \left[\frac{x^3}{3}\right]_{1}^{\frac{3}{2}} = 1 - \frac{1}{2} + \frac{27}{8} \cdot \frac{1}{3} - \frac{1}{3} = \frac{31}{24}$$

Příklad 2 (d) $\int_{-2}^{1} e^{-|x|} \mathrm{d}x$.

Platí $f(x) := e^{-|x|} = \begin{cases} e^{x}, & x \in (-\infty, 0] \\ e^{-x}, & x \in (0, \infty) \end{cases}$ je spojitá funkce na $\mathbb{R}$. Použijeme aditivitu integrálu.

$$\int_{-2}^{1} e^{-|x|} \mathrm{d}x = \int_{-2}^{0} e^{x} \mathrm{d}x + \int_{0}^{1} e^{-x} \mathrm{d}x = [e^{x}]_{-2}^{0} + \left[-e^{-x}\right]_{0}^{1} = 1 - \frac{1}{e^2} - \frac{1}{e} + 1 = \frac{2e^2 - e - 1}{e^2}$$

Příklad 2 (e) $\int_{-\frac{17}{12}\pi}^{-\frac{3}{4}\pi} |\sin x| \mathrm{d}x$.

$f(x) := |\sin x| = \begin{cases} \sin x, & x \in [2k\pi, (2k+1)\pi], k \in \mathbb{Z} \\ -\sin x, & x \in ((2k+1)\pi, (2k+2)\pi), k \in \mathbb{Z} \end{cases}$

Jde o spojitou funkci, tedy lze použít aditivu integrálu, vyjde-li nám definovaný výraz.

Uvědomme si, kde na reálné ose jsou $-\frac{17}{12}\pi$ a $-\frac{3}{4}\pi$. Platí

$$-2\pi < -\frac{17}{12}\pi < -\pi < -\frac{3}{4}\pi < 0.$$

Nyní si můžeme uvědomit, že

$$|\sin x| = \begin{cases} \sin x, & x \in \left(-\frac{17}{12}\pi, -\pi\right), \\ -\sin x, & x \in \left(-\pi, -\frac{3}{4}\pi\right) \end{cases}$$

$$\int_{-\frac{17}{12}\pi}^{-\frac{3}{4}\pi} |\sin x| \mathrm{d}x = \int_{-\frac{17}{12}\pi}^{-\pi} \sin x \mathrm{d}x + \int_{-\pi}^{-\frac{3}{4}\pi} -\sin x \mathrm{d}x = [-\cos x]_{-\frac{17}{12}\pi}^{-\pi} + [\cos x]_{-\pi}^{-\frac{3}{4}\pi} =$$
$$= -\cos(-\pi) + \cos\left(-\frac{17}{12}\pi\right) + \cos\left(-\frac{3}{4}\pi\right) - \cos(-\pi) = \cos\left(\frac{17}{12}\pi\right) + \cos\left(\frac{3}{4}\pi\right) + 2$$

Příklad 2 (f) $\int_{-\frac{\pi}{3}}^{\frac{7}{3}\pi} \frac{1}{1+\sin^2 x}\mathrm{d}x$.

Pro racionální funkci $R(x,y) = \frac{1}{1+x^2}$ platí $R(-\sin x, -\cos x) = R(\sin x, \cos x)$. Budeme tedy volit substituci $t = \tan x$ pro výpočet primitivní funkce na intervalech $\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right), k \in \mathbb{Z}$.

$$-\frac{\pi}{3} < \frac{\pi}{2} < \frac{3}{2}\pi < \frac{7}{3}\pi$$

Integrovaná funkce je v bodě $\frac{\pi}{2}$ spojitá, lze proto použít aditivitu integrálu.

V substituci níže si uvědomme, že funkce $x \mapsto \tan x$ je na intervalech $\left(-\frac{\pi}{3}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3}{2}\pi\right), \left(\frac{3}{2}\pi, \frac{7}{3}\pi\right)$ ryze monotónní, analogicky pro $t \mapsto t\sqrt{2}$ na příslušných intervalech.

$$\int_{-\frac{\pi}{3}}^{\frac{7}{3}\pi} \frac{1}{1+\sin^2 x}\mathrm{d}x = \int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1+\sin^2 x}\mathrm{d}x + \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{1}{1+\sin^2 x}\mathrm{d}x + \int_{\frac{3}{2}\pi}^{\frac{7}{3}\pi} \frac{1}{1+\sin^2 x}\mathrm{d}x =$$
$$= \left| \begin{array}{cc} t = \tan x, & \mathrm{d}x = \frac{1}{1+t^2}\mathrm{d}t \\ -\frac{\pi}{3} \to \tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}, & \frac{\pi}{2}- \to \infty \\ \frac{\pi}{2}+ \to -\infty, & \frac{3}{2}\pi- \to \infty \\ \frac{3}{2}\pi+ \to -\infty & \frac{7}{3}\pi \to \tan\left(\frac{7}{3}\pi\right) = \sqrt{3} \end{array} \right| =$$
$$= \int_{-\sqrt{3}}^{\infty} \frac{1}{2t^2+1}\mathrm{d}t + \int_{-\infty}^{\infty} \frac{1}{2t^2+1}\mathrm{d}t + \int_{-\infty}^{\sqrt{3}} \frac{1}{2t^2+1}\mathrm{d}t = \left|u = t\sqrt{2}, \mathrm{d}u = \sqrt{2}\mathrm{d}t\right| =$$
$$= \frac{1}{\sqrt{2}} \int_{-\sqrt{6}}^{\infty} \frac{1}{u^2+1}\mathrm{d}u + \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{1}{1+u^2}\mathrm{d}u + \frac{1}{\sqrt{2}} \int_{-\infty}^{\sqrt{6}} \frac{1}{1+u^2}\mathrm{d}u =$$
$$= \frac{1}{\sqrt{2}} [\arctan u]_{-\sqrt{6}}^{\infty} + \frac{1}{\sqrt{2}} [\arctan u]_{-\infty}^{\infty} + \frac{1}{\sqrt{2}} [\arctan u]_{-\infty}^{\sqrt{6}} =$$
$$= \frac{1}{\sqrt{2}} \left(\lim_{u\to\infty} \arctan u - \arctan(-\sqrt{6}) + \lim_{u\to\infty} \arctan u - \lim_{u\to-\infty} \arctan u + \arctan(\sqrt{6}) - \lim_{u\to-\infty} \arctan u \right) =$$
$$= \frac{1}{\sqrt{2}} \left(2\pi + 2\arctan(\sqrt{6})\right) = \sqrt{2}\left(\pi + \arctan(\sqrt{6})\right)$$

■

Příklad 2 (g) $\int_{-\frac{3}{2}\pi}^{0} \frac{1}{\sin x + \cos x + 2}\mathrm{d}x$.

Pro $R(x,y) := \frac{1}{x+y+2}$ neplatí ani jedno z následujícího: $R(-\sin x, \cos x) = R(\sin x, \cos x)$, $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, $R(-\sin x, -\cos x) = R(\sin x, \cos x)$. Proto budeme volit substituci $t = \tan \frac{x}{2}$. Tu lze provést pouze na intervalech $(-\pi + k2\pi, \pi + k2\pi)\,, k \in \mathbb{Z}$. Budeme pak tedy muset v bodech $\pi + k2\pi$ integrál rozdělit.

Integrovaná funkce je v bodě $-\pi$ spojitá, lze proto použít aditivitu integrálu.

V substituci níže si uvědomme, že funkce $x \mapsto \tan \frac{x}{2}$ je na intervalech $\left(-\frac{3}{2}\pi, -\pi\right), (-\pi, 0)$ ryze monotónní, podobně pro $t \mapsto \frac{t+1}{\sqrt{2}}$.

$$
\begin{aligned}
&\int_{-\frac{3}{2}\pi}^{0} \frac{1}{\sin x + \cos x + 2}\mathrm{d}x = \int_{-\frac{3}{2}\pi}^{-\pi} \frac{1}{\sin x + \cos x + 2}\mathrm{d}x + \int_{-\pi}^{0} \frac{1}{\sin x + \cos x + 2}\mathrm{d}x = \\
&= \left| \begin{matrix} t = \tan\frac{x}{2}, & \mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t \\ -\frac{3}{2}\pi+ \to \tan\left(-\frac{3}{4}\pi\right) = 1, & -\pi- \to \infty \\ -\pi+ \to -\infty, & 0 \to 0 \end{matrix} \right| = \\
&= \int_{1}^{\infty} \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 2} \cdot \frac{2}{1+t^2}\mathrm{d}t + \int_{-\infty}^{0} \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 2} \cdot \frac{2}{1+t^2}\mathrm{d}t = \\
&\overset{(☎)}{=} \int_{1}^{\infty} \frac{1}{\left(\frac{t+1}{\sqrt{2}}\right)^2 + 1}\mathrm{d}t + \int_{-\infty}^{0} \frac{1}{\left(\frac{t+1}{\sqrt{2}}\right)^2 + 1}\mathrm{d}t = \left| \begin{matrix} u = \frac{t+1}{\sqrt{2}}, & \mathrm{d}u = \frac{1}{\sqrt{2}}\mathrm{d}t \\ 1 \to \frac{2}{\sqrt{2}} = \sqrt{2}, & \infty \to \infty \\ -\infty \to -\infty, & 0 \to \frac{1}{\sqrt{2}} \end{matrix} \right| = \\
&\overset{\text{lin.}}{=} \sqrt{2}\int_{\sqrt{2}}^{\infty} \frac{1}{u^2+1}\mathrm{d}u + \sqrt{2}\int_{-\infty}^{\frac{1}{\sqrt{2}}} \frac{1}{u^2+1}\mathrm{d}u = \\
&= \sqrt{2}\left[\arctan u\right]_{\sqrt{2}}^{\infty} + \sqrt{2}\left[\arctan u\right]_{-\infty}^{\frac{1}{\sqrt{2}}} = \\
&= \sqrt{2}\left(\lim_{u\to\infty} \arctan u - \arctan(\sqrt{2}) + \arctan\left(\frac{1}{\sqrt{2}}\right) - \lim_{u\to-\infty} \arctan u\right) = \\
&= \sqrt{2}\left(\frac{\pi}{2} - \arctan(\sqrt{2}) + \arctan\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{2}\right) = \sqrt{2}\left(\pi + \arctan\left(\frac{1}{\sqrt{2}}\right) - \arctan(\sqrt{2})\right)
\end{aligned}
$$

$$
\begin{aligned}
(☎): \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 2} \cdot \frac{2}{1+t^2} &= \frac{1+t^2}{2t + 1 - t^2 + 2 + 2t^2} \cdot \frac{2}{1+t^2} = \frac{2}{t^2 + 2t + 3} = \\
&= 2 \cdot \frac{1}{(t+1)^2 + 2} = \frac{1}{\left(\frac{t+1}{\sqrt{2}}\right)^2 + 1}
\end{aligned}
$$

Příklad 2 (h) $\int_{-\pi}^{0} \frac{1}{3\cos^2 x + \sin 2x + 1}\mathrm{d}x$.

Pro $R(x,y) = \frac{1}{3y^2 + 2xy + 1}$ platí následující: $R(-\sin x, -\cos x) = R(\sin x, \cos x)$. Budeme tedy volit substituci $t = \tan x$, což lze provést pouze na intervalech $\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right), k \in \mathbb{Z}$.

Integrovaná funkce je v bodě $-\frac{\pi}{2}$ spojitá, lze proto použít aditivitu integrálu.

V substituci níže si uvědomme, že funkce $x \mapsto \tan x$ je na intervalu $\left(-\pi, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, 0\right)$ ryze monotónní, analogicky pro $t \mapsto \frac{t+1}{\sqrt{3}}$.

$$\int_{-\pi}^{0} \frac{1}{3\cos^2 x + \sin 2x + 1}\mathrm{d}x = \int_{-\pi}^{-\frac{\pi}{2}} \frac{1}{3\cos^2 x + 2\sin x \cos x + 1}\mathrm{d}x + \int_{-\frac{\pi}{2}}^{0} \frac{1}{3\cos^2 x + 2\sin x \cos x + 1}\mathrm{d}x =$$

$$= \left| \begin{matrix} t = \tan x, & \mathrm{d}x = \frac{1}{1+t^2}\mathrm{d}t \\ -\pi \to \tan(-\pi) = 0, & -\frac{pi}{2}- \to \infty \\ -\frac{\pi}{2}+ \to -\infty, & 0 \to 0 \end{matrix} \right| =$$

$$\int_{0}^{\infty} \frac{1}{3\frac{1}{1+t^2} + 2\frac{t}{1+t^2} + 1} \cdot \frac{1}{1+t^2}\mathrm{d}t + \int_{-\infty}^{0} \frac{1}{3\frac{1}{1+t^2} + 2\frac{t}{1+t^2} + 1} \cdot \frac{1}{1+t^2}\mathrm{d}t =$$

$$\overset{(\text{✈})}{=} \frac{1}{3}\int_{0}^{\infty} \frac{1}{\left(\frac{t+1}{\sqrt{3}}\right)^2 + 1}\mathrm{d}t + \frac{1}{3}\int_{-\infty}^{0} \frac{1}{\left(\frac{t+1}{\sqrt{3}}\right)^2 + 1}\mathrm{d}t = \left| \begin{matrix} u = \frac{t+1}{\sqrt{3}}, & \mathrm{d}u = \frac{1}{\sqrt{3}}\mathrm{d}t \\ 0 \to \frac{1}{\sqrt{3}}, & \infty \to \infty \\ -\infty \to -\infty, & 0 \to \frac{1}{\sqrt{3}} \end{matrix} \right| =$$

$$= \frac{1}{\sqrt{3}}\left([\arctan u]_0^{\infty} + [\arctan u]_{-\infty}^{0} \right) = \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - 0 + 0 + \frac{\pi}{2} \right) = \frac{\pi}{\sqrt{3}}$$

$$(\text{✈}): \frac{1}{3\frac{1}{1+t^2} + 2\frac{t}{1+t^2} + 1} \cdot \frac{1}{1+t^2} = \frac{1+t^2}{3 + 2t + 1 + t^2} \cdot \frac{1}{1+t^2} = \frac{1}{t^2 + 2t + 4} = \frac{1}{(t+1)^2 + 3} = \frac{1}{3} \cdot \frac{1}{\left(\frac{t+1}{\sqrt{3}}\right)^2 + 1}$$

Příklad 2 (i) $\int_{\frac{\pi}{3}}^{\pi} \frac{1}{6\cos^2 x + 4\sin x \cos x + \sin^2 x}\mathrm{d}x$.

Pro $R(x,y) = \frac{1}{6y^2+4xy+x^2}$ platí $R(-\sin x, -\cos x) = R(\sin x, \cos x)$. Budeme tedy volit substituci $t = \tan x$, což lze provést pouze na intervalech $\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right), k \in \mathbb{Z}$.

Integrovaná funkce je v bodě $\frac{\pi}{2}$ spojitá, lze proto použít aditivitu integrálu.

V substituci níže si uvědomme, že funkce $x \mapsto \tan x$ je na intervalu $\left(\frac{\pi}{3}, \frac{\pi}{2}\right), \left(\frac{\pi}{2,\pi}\right)$ ryze monotónní, analogicky pro $t \mapsto \frac{t+2}{\sqrt{2}}$.

$$\int_{\frac{\pi}{3}}^{\pi} \frac{1}{6\cos^2 x + 4\sin x\cos x + \sin^2 x}\mathrm{d}x = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{6\cos^2 x + 4\sin x\cos x + \sin^2 x}\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} \text{-//-}\mathrm{d}x =$$

$$= \left| \begin{array}{cc} t = \tan x, & \mathrm{d}x = \frac{1}{1+t^2}\mathrm{d}t \\ \frac{\pi}{3} \to \tan\frac{\pi}{3} = \sqrt{3}, & \frac{\pi}{2}- \to \infty \\ \frac{\pi}{2}+ \to -\infty, & \pi \to \tan(\pi) = 0 \end{array} \right| =$$

$$= \int_{\sqrt{3}}^{\infty} \frac{1}{6\frac{1}{1+t^2} + 4\frac{t}{1+t^2} + \frac{t^2}{1+t^2}} \cdot \frac{1}{t^2+1}\mathrm{d}t + \int_{-\infty}^{0} \frac{1}{6\frac{1}{1+t^2} + 4\frac{t}{1+t^2} + \frac{t^2}{1+t^2}} \cdot \frac{1}{t^2+1}\mathrm{d}t =$$

$$\overset{(✍)}{=} \frac{1}{2}\int_{\sqrt{3}}^{\infty} \frac{1}{\left(\frac{t+2}{\sqrt{2}}\right)^2+1}\mathrm{d}t + \frac{1}{2}\int_{-\infty}^{0} \frac{1}{\left(\frac{t+2}{\sqrt{2}}\right)^2+1}\mathrm{d}t = \left| \begin{array}{cc} u = \frac{t+2}{\sqrt{2}}, & \mathrm{d}u = \frac{1}{\sqrt{2}}\mathrm{d}t \\ \sqrt{3} \to \frac{2+\sqrt{3}}{\sqrt{2}}, & \infty \to \infty \\ -\infty \to -\infty, & 0 \to \frac{2}{\sqrt{2}} \end{array} \right| =$$

$$= \frac{1}{\sqrt{2}}\left([\arctan u]_{\frac{2+\sqrt{3}}{\sqrt{2}}}^{\infty} + [\arctan u]_{-\infty}^{\frac{2}{\sqrt{2}}} \right) = \frac{1}{\sqrt{2}}\left(\frac{\pi}{2} - \arctan\frac{2+\sqrt{3}}{\sqrt{2}} + \arctan\frac{2}{\sqrt{2}} + \frac{\pi}{2} \right) =$$

$$= \frac{1}{\sqrt{2}}\left(\pi - \arctan\frac{2+\sqrt{3}}{\sqrt{2}} + \arctan\sqrt{2} \right)$$

$$(✍): \frac{1}{6\frac{1}{1+t^2} + 4\frac{t}{1+t^2} + \frac{t^2}{1+t^2}} \cdot \frac{1}{t^2+1} = \frac{1+t^2}{6+4t+t^2} \cdot \frac{1}{t^2+1} = \frac{1}{t^2+4t+6}\mathrm{d}t = \frac{1}{(t+2)^2+2} =$$

$$= \frac{1}{2} \cdot \frac{1}{\left(\frac{t+2}{\sqrt{2}}\right)^2+1}$$

Příklad 3 (a) $\int \frac{\tan x}{6+11\cos x+6\cos^2 x+\cos^3 x}\mathrm{d}x$

$$\int \frac{\tan x}{6+11\cos x+6\cos^2 x+\cos^3 x}\mathrm{d}x = |t = \cos x, \mathrm{d}t = -\sin x\mathrm{d}x| = \int \frac{-\frac{1}{t}}{t^3+6t^2+11t+6}\mathrm{d}t =$$

$$= \int \frac{-1}{t(t+1)(t+2)(t+3)}\mathrm{d}t = \int \frac{\frac{1}{2}}{t+1} - \frac{\frac{1}{2}}{t+2} + \frac{\frac{1}{6}}{t+3} - \frac{\frac{1}{6}}{t}\mathrm{d}t =$$

$$\overset{c}{=} \frac{1}{2}\log|t+1| - \frac{1}{2}\log|t+2| + \frac{1}{6}\log|t+3| - \frac{1}{6}\log|t| =$$

$$\overset{c}{=} \frac{1}{2}\log|\cos x+1| - \frac{1}{2}\log|\cos x+2| + \frac{1}{6}\log|\cos x+3| - \frac{1}{6}\log|\cos x|$$

Příklad 3(m) $\int_2^{\infty} \frac{1}{x} \cdot \frac{1}{\log^3 x+2\log x}\mathrm{d}x$

$$\int_2^{\infty} \frac{1}{x} \cdot \frac{1}{\log^3 x+2\log x}\mathrm{d}x = \left| y = \log x, \mathrm{d}y = \frac{1}{x}\mathrm{d}x, 2 \to \log 2, \infty \to \infty \right| = \int_{\log 2}^{\infty} \frac{1}{y^3+2y}\mathrm{d}y$$

Parciální zlomky:

$$
\begin{aligned}
&\frac{1}{y^3+2y} = \frac{1}{y(y^2+2)} = \frac{A}{y} + \frac{By+C}{y^2+2} \\
&1 = Ay^2 + 2A + By^2 + Cy = y^2(A+C) + Dy + 2A \implies \qquad 0 = A+C \\
&0 = D \\
&1 = 2A
\end{aligned}
$$

Tedy $\frac{1}{y^3+2y} = \frac{\frac{1}{2}}{y} + \frac{-\frac{1}{2}y}{y^2+2}$.

$$
\int \frac{\frac{1}{2}}{y} + \frac{-\frac{1}{2}y}{y^2+2}\mathrm{d}y \overset{\text{lin.}}{=} \frac{1}{2}\int \frac{1}{y}\mathrm{d}y - \frac{1}{4}\int \frac{2y}{y^2+2}\mathrm{d}y = \left|z = y^2+2, \mathrm{d}z = 2y\mathrm{d}y\right| = \frac{1}{2}\log|y| - \frac{1}{4}\int \frac{1}{z}\mathrm{d}z = \\
\overset{\text{c}}{=} \frac{1}{2}\log|y| - \frac{1}{4}\log(y^2+2)
$$

$$
\begin{aligned}
&\int_{\log 2}^{\infty} \frac{1}{y^3+2y}\mathrm{d}y = \left[\frac{1}{2}\log|y| - \frac{1}{4}\log(y^2+2)\right]_{\log 2}^{\infty} = \left[\log\frac{\sqrt{y}}{\sqrt[4]{y^2+2}}\right]_{\log 2}^{\infty} = \\
&= \lim_{y\to\infty}\log\frac{\sqrt{y}}{\sqrt[4]{y^2+2}} - \lim_{y\to\log 2-}\log\frac{\sqrt{y}}{\sqrt[4]{y^2+2}} = 0 - \lim_{y\to\log 2-}\frac{1}{4}\log\frac{y^2}{y^2+2} = \frac{1}{4}\log\frac{\log^2 2}{\log^2 2+2}
\end{aligned}
$$